Nonlinear Acoustic Propagation

In the previous derivation of the acoustic wave equation we considered only linear disturbances. Let's look briefly at what happens when you include nonlinear terms.
In this derivation we are going to consider Nonlinear Propagation Conditions for the:

a) Nonlinear Equation of State
b) Nonlinear Wave Equation

<u>Nonlinear Equation of State</u>

Assume that pressure is a function of density, $P = P(r)$ where $P = p_0 + p$ and $r = r_0 + r'$, the ambient pressure (density) plus the perturbed pressure (density).

As before, we expand $p_0 + p = P(r_0 + r')$ in a Taylor series about equilibrium density, that is (see Eq 5.2.4),

$$p_0 + p = P(r_0 + r') = P(r_0) + r'P'(r_0) + \frac{r'^2}{2!}P''(r_0) + \ldots = p_0 + r'P'(r_0) + \frac{r'^2}{2!}P''(r_0) + \ldots$$

Subtracting p_0 from both sides leaves us with an expression for the instantaneous pressure

$$p = r'P'(r_0) + \frac{r'^2}{2!}P''(r_0) + \ldots$$

Note that for the <u>linearized</u> approximation we assume that the $\frac{r_0^2}{2!}P''(r_0) + \ldots$ terms (the nonlinear terms) are negligible and therefore discount them in the subsequent derivations. When we did this, of course, we showed that:

$$p = r'P'(r_0) = r'\left(\frac{\P P}{\P r}\right)_0 = r_0 c_0^2$$

Now, if the previously neglected terms are <u>retained</u>,

$$p = r'P'(r_0) + \frac{r'^2}{2!}P''(r_0) + \ldots = (r - r_0)P'(r_0) + \frac{(r - r_0)^2}{2!}P''(r_0) + \ldots$$

$$= r_0 P'(r_0)\left(\frac{r - r_0}{r_0}\right) + \frac{r_0^2 P''(r_0)}{2!}\left(\frac{r - r_0}{r_0}\right)^2 + \ldots = As + \frac{B}{2}s^2 \qquad \textbf{Eq A}$$

where

$$A = r_0 P'(r_0) = r_0 c_0^2$$
$$B = r_0^2 P''(r_0)$$

The ratio $\frac{B}{A}$ is a significant parameter in expressing the nonlinear properties of the medium where

$$\frac{B}{A} = \frac{r_0^2 P''(r_0)}{r_0 c_0^2} = \frac{r_0}{c_0^2}\left(\frac{\P^2 p}{\P r^2}\right)_{s, r_0}$$

Looking at an adiabatic process for a perfect gas we know that, $P = p_0 \left(\frac{r}{r_0} \right)^g$, and expanding in terms of the condensation s yields:

$$p = g\, p_0 s + \frac{g\,(g-1)\, p_0}{2} s^2 + \ldots \qquad \textbf{Eq B}$$

Comparing the coefficients of Eqs A and B yields

$$A = g\, p_0$$

$$B = g\,(g-1)\, p_0$$

and

$$\frac{B}{A} = \frac{g\,(g-1)\, p_0}{g\, p_0} = g - 1$$

<u>Nonlinear Wave Equation</u>

In the linear derivation of the wave equation we derived the following relations:

$$r_0 = r\left(1 + \frac{\partial x_x}{\partial x}\right) \qquad \textbf{Eq C}$$

$$-\nabla p = r_0 \frac{\partial u}{dt} = r_0 \frac{\partial^2 x}{dt^2} \qquad \text{(Linear Euler's Equation)}$$

$$p = r_0 c^2 s = c^2 \left(r - r_0\right) \qquad \text{(Equation of State).}$$

These combined gave us the Linear Wave Equation. If we take the derivative of the Equation of State with respect to x (assuming a 1D case again) we have

$$\frac{\partial p}{\partial x} = c^2 \frac{\partial r}{\partial x}$$

Rearranging **Eq C** gives

$$r = r_0 \left(1 + \frac{\partial x_x}{\partial x}\right)^{-1}$$

so that

$$\frac{\partial r}{\partial x} = r_0 \frac{\partial}{\partial x}\left(1 + \frac{\partial x_x}{\partial x}\right)^{-1} = -r_0 \left(1 + \frac{\partial x_x}{\partial x}\right)^{-2} \frac{\partial^2 x_x}{\partial x^2}.$$

Therefore,

$$\frac{\partial p}{\partial x} = -c^2 r_0 \left(1 + \frac{\partial x_x}{\partial x}\right)^{-2} \frac{\partial^2 x_x}{\partial x^2}$$

and if we substitute this into the Euler's Equation we get

$$c^2 r_0 \left(1 + \frac{\partial x_x}{\partial x}\right)^{-2} \frac{\partial^2 x_x}{\partial x^2} = r_0 \frac{\partial^2 x_x}{\partial t^2}$$

or

$$\frac{\partial^2 x_x}{\partial t^2} = \frac{c^2}{\left(1+\frac{\partial x_x}{\partial x}\right)^2} \frac{\partial^2 x_x}{\partial x^2} .$$

To bring the nonlinear parameters $\frac{B}{A}$ into the equation, we look further at the relations between the speed of sound and the pressure/density relations (Equation of state).

$$c^2 = \left(\frac{\partial P}{\partial r}\right) = \frac{\partial}{\partial r}\left(p_0\left(\frac{r}{r_0}\right)^g\right) = \frac{g\, p_0}{r_0}\left(\frac{r}{r_0}\right)^{g-1} = c_0^2\left(\frac{r}{r_0}\right)^{g-1} = \frac{c_0^2}{\left(1+\frac{\P x}{\P\, x}\right)^{g-1}}$$

$$\frac{\partial^2 x}{\partial t^2} = \frac{c^2}{\left(1+\frac{\partial x}{\partial x}\right)^2}\frac{\partial^2 x}{\partial x^2} = \frac{c_0^2}{\left(1+\frac{\partial x}{\partial x}\right)^{g+1}}\frac{\partial^2 x}{\partial x^2}$$

$$\frac{\partial^2 x}{\partial t^2} = \frac{c_0^2}{\left(1+\frac{\partial x}{\partial x}\right)^{B/A+2}}\frac{\partial^2 x}{\partial x^2}$$ (Basic Nonlinear Wave Equation)

This expression can serve as the basic nonlinear equation and a starting point for most nonlinear acoustic applications. All you have to know are the $\frac{B}{A}$ values for the particular medium of interest to be able predict certain nonlinear behaviors.

Observe that if we start from **Eq B**

$$p = g\, p_0 s + \frac{g\,(g-1)\,p_0}{2}s^2 = (r - r_0)\left[\frac{g\, p_0}{r_0} + \frac{g\,(g-1)\,p_0}{2}\frac{(r-r_0)}{r_0^2}\right] = (r - r_0)\left[\frac{g\, p_0}{r_0} + \frac{g\, p_0}{r_0}\frac{(g-1)\,r'}{2\,r_0}\right]$$

$$= (r - r_0)\left(\frac{g\, p_0}{r_0}\right)\left[1+\frac{(g-1)\,r'}{2\,r_0}\right]$$

For adiabatic perfect gas we showed that

$$c_0^2 = \frac{g\, p_0}{r_0}$$

so

$$p = (r - r_0)\,c_0^2\left[1+\frac{(g-1)\,r'}{2\,r_0}\right] = (r - r_o)\,c^2$$

where

$$c^2 = c_0^2\left[1+\frac{(g-1)\,r'}{2\,r_0}\right]$$

Now if we approximate: $r - r_0 = r' \approx \frac{p}{c_0^2} = \frac{p}{\left(\frac{g\,p_0}{r_0}\right)} = \frac{p\,r_0}{g\,p_0}$

then

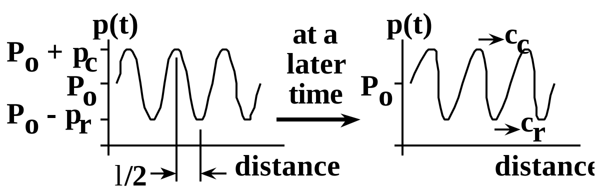
$$c^2 = c_0^2\left[1 + \frac{(g-1)\,r'}{2r_0}\right] = c_0^2\left[1 + \frac{(g-1)\left(\frac{p\,r_0}{g\,p_0}\right)}{2r_0}\right] = c_0^2\left[1 + \frac{(g-1)}{2g}\frac{p}{p_0}\right] = c_0^2\left[1 + \frac{B/A}{2g}\frac{p}{p_0}\right]$$

From $c^2 = c_0^2\left[1 + \frac{B/A}{2g}\frac{p}{p_0}\right]$ observe that for air where $\gamma = 1.4$

$$\frac{B/A}{2g} = \frac{0.4}{2\cdot 1.4} \approx 0.14$$

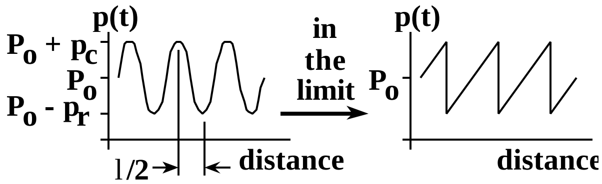
$$c^2 = c_0^2\left[1 + 0.14\,\frac{p}{p_0}\right] \quad \text{(for air where } \gamma = 1.4\text{)}$$

Physically, what $c^2 = c_o^2\left[1 + \frac{B/A}{2\gamma}\frac{p}{P_o}\right]$ means is that speed is dependent upon the amplitude of the pressure disturbance. The speed is greater in regions of compression (+ pressure) and lower in regions of rarefaction (- pressure), that is, $c_c > c_r$ (both c_c and c_r are positive numbers).



(both p_c and p_r are positive numbers)

Thus, a region of compression (positive particle velocity) has $c > c_0$ and a region of rarefaction has $c < c_0$, so that the wave becomes distorted as it travels. Since the wave becomes distorted it no longer has a single frequency component. Instead some energy is transferred from the fundamental frequency that was generated at the source to higher harmonics (integer multiples of the fundamental frequency). In the limit where the slope becomes -8, a shock wave is formed. This occurs when the crest catches up with the trough. This is often called an N-wave because:



Thus the speed depends upon $\frac{\partial x_x}{\partial x}$ and therefore position on the waveform.

A useful parameter to determine just how significant this nonlinear distortion might be is the distance, *D*, at which a shock wave is formed. The shock wave formation distance is defined as the distance where the slope of the waveform at its zero crossing becomes -8 in a lossless (no attenuation of energy in a linearly propagating wave) medium. This marks the distance where a shock wave starts to form and where nonlinear effects are of obvious importance.
To estimate the distance D the wave propagates to form a fully-developed shock wave, we consider the time it takes for the crest to travel an increased distance λ/2 to catch up with the trough, that is,

$$t = \frac{D + \frac{l}{2}}{c_c} = \frac{D}{c_r}$$

rearranging to solve for *D*

$$\frac{c_c}{D + \frac{l}{2}} = \frac{c_r}{D}$$

$$(c_c - c_r)D = c_r \frac{\lambda}{2}$$

Therefore,

$$D = \frac{c_r \frac{l}{2}}{c_c - c_r}$$

This can be slightly approximated:
Assume $l << D$ then

$$(c_c - c_r) = \frac{c_r l}{2D} = \frac{c_r}{2}\left(\frac{l}{D}\right) << 1$$

$$\rightarrow c_c \approx c_r \approx c_0$$

Therefore,

$$D = \frac{c_0 \frac{l}{2}}{c_c - c_r} \qquad \textbf{Eq A}$$

From this expression, the various speed quantities must be calculated in order to determine *D.*
Another expression to determine D can be found by starting with

$$c^2 = c_0^2\left[1 + \frac{(g-1)}{2g}\frac{p}{p_0}\right] = c_0^2\left[1 + \frac{B/A}{2g}\frac{p}{p_0}\right]$$

$$c = c_0\sqrt{1 + \frac{B/A}{2g}\frac{p}{p_0}}$$

Expanding the term under the radical in a series gives:

$$c = c_0\left[1 + \frac{1}{2}\frac{B/A}{2g}\frac{p}{p_0} + \ldots\right]$$

Thus (both p_c and p_r are positive numbers),

$$c_c = c_0\left[1+\frac{1}{2}\frac{B/A}{2g}\frac{p_c}{p_0}\right]$$

$$c_r = c_0\left[1-\frac{1}{2}\frac{B/A}{2g}\frac{p_r}{p_0}\right]$$

Substituting c_c and c_r into Eq A, $D=\dfrac{c_0\frac{l}{2}}{c_c-c_r}$, yields:

$$D=\frac{c_0\frac{l}{2}}{c_0\left[1+\frac{1}{2}\frac{B/A}{2g}\frac{p_c}{p_0}\right]-c_0\left[1-\frac{1}{2}\frac{B/A}{2g}\frac{p_r}{p_0}\right]}=\frac{\frac{l}{2}}{1+\frac{1}{2}\frac{B/A}{2g}\frac{p_c}{p_0}-1+\frac{1}{2}\frac{B/A}{2g}\frac{p_r}{p_0}}=\frac{\frac{l}{2}}{\frac{B/A}{4g\,p_0}(p_c+p_r)}$$

Therefore,

$$D=\frac{2gl}{B/A}\frac{p_0}{(p_c+p_r)}$$

****************************** Example 5.11 ******************************

Consider a 1 kHz source in air at 1 atmosphere and 20°C at a SPL = 100 dB. At what distance is a fully-developed shock wave formed? What about SPL = 120 dB and 135 dB?

ANSWER:

- If SPL = 100 dB, $100 = 20\log\left(\frac{p_e}{p_{ref}}\right)$, $\frac{p_e}{p_{ref}} = 10^5 \rightarrow p = 2.828\,Pa$ (peak), $p_c = 2.828\,Pa$, $p_r = 2.828\,Pa$ and $l = \frac{c_0}{f} = \frac{343\text{m/s}}{1000\text{Hz}} = 0.343\text{m}$. $D=\frac{2gl}{B/A}\frac{p_o}{(p_c+p_r)}$ $= \frac{2(1.402)(0.343\text{m})}{0.402}\frac{101{,}330\text{Pa}}{5.656\text{Pa}}$ = 42.9 km

- If SPL = 120 dB, D = 4.29 km
- If SPL = 135 dB, D = 762 m

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****************************** Example 5.12 ******************************

As seen from the previous example, in air at 20°C, sound propagation is nonlinear at a SPL of 135 dB. Using *linear* approximations, determine the particle displacement of air under these conditions and frequency of 1 kHz.

ANSWER: $p = p_{ref} \cdot 10^{SPL/20} = 20\ \mu\text{Pa(rms)} \cdot 10^{135/20}$ = 112.47 Pa (rms).

$x_o = \dfrac{p}{r_o c w}$ = 43 µm (rms)

Note that this is approximately *7 orders of magnitude* larger than the particle displacement corresponding to 0 dB.
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Examples of values for *D* in water at 20°C are provided in the following table.

Intensity (W/cm^2)	Acoustic Pressure (MPa)	Particle Speed (m/s)	Acoustic Mach Number	*D* @ 1 kHz (m)	*D* @ 1 MHz (m)
0.1	0.0544	0.0368	2.48 x 10^{-5}	2,710	2.71
1	0.172	0.116	7.85 x 10^{-5}	860	0.86
10	0.544	0.368	2.48 x 10^{-4}	270	0.27

A final parameter that is of use when dealing with real-world materials that have some loss is the Goldberg number given by

$$\Gamma = \frac{M b}{a/k} = \frac{1}{a L_d},$$

where a is the attenuation coefficient that will be examined later when we talk about loss in Chapter 8. The Goldberg number can be thought of as the ratio of the measure of the strength of the nonlinear effect (Mb) to the measure of attenuation over a distance of one wavelength (a/k). For $\Gamma << 1$ (very lossy material such as viscous oils or most biological tissues) the wave decays before significant nonlinear distortion and for $\Gamma >> 1$ (low loss material such as water) shock waves form before the wave has attenuated appreciably.